# Pricing Barrier Options with a Volatility Term Structure

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## Abstract

We study the use of Monte Carlo techniques to price barrier options in the case where the observation points are discrete, including several methods for reducing the variance of the final result. In particular, we also compute hedge parameters delta and gamma and study the effect on the pricing of barrier options of different volatility structures.

## I. INTRODUCTION

Monte Carlo simulation has a widespread application in many different areas, such as Physics, Engineering and also in Economics. It employs random variables in order to solve a specific problem. The Monte Carlo method is based on two general principles: the Central Limit Theorem and the Law of Large Numbers. It was first used for option valuation problems by Boyle [1]. It is particularly useful because of its flexibility, allowing generalization for more difficult situations, including basket options, models with stochastic volatility and interest rates, where closed form analytical solutions do not exist. The main disadvantage of the method is the necessity of large computing time in order to achieve trustable results, *i. e.* results with low variance.

In this paper we will apply Monte Carlo techniques in order to valuate barrier options with discrete observation times, in which case no analytical exact solution is available. We will study the efficiency of various variance reduction techniques, such as antithetic variables, control variables and quasi Monte Carlo and compute the hedge parameters $\Delta$ and $\Gamma$. We will also analyse the impact of volatility changes on the pricing of these options.

In order to have a feeling for the quality of the Monte Carlo result, we shall first compare it with the known results of a plain vanilla option, which has a closed form pricing formula given by the well-known Black-Scholes formula.

## II. THE MONTE CARLO METHOD

The Monte Carlo method consists of generating a large number of possible outcomes of the price of the option weighted by a given probability for each outcome and estimating its real value by taking the appropriate average.

The fair value of an option today written on an underlying asset with price $S$ is given by [2]:

$$V_0 = E_Q \left( \exp \left( - \int_0^T r_u du \right) X(S) \right) \tag{1}$$

where $E_Q$ denotes an expectation value in a risk-free measure where the discounted stock price $S$ is a martingale process, $r_u$ is the risk-free interest rate at the instant $u$, $T$ is the expiration date and $X(S)$ is the price of the option as a function of the process $S$.

In what follows we will concentrate in the case where the risk free interest rate is a constant $r$, and equation 1 reduces to:

$$V_0 = \exp(-rT) E_Q (X(S)) \tag{2}$$

The starting point of the Monte Carlo method is to generate possible trajectories for the price of the underlying asset $S$ in the risk free measure. This is accomplished by assuming that the process follows an exponential brownian motion described by the stochastic differential equation:

$$\frac{dS_t}{S_t} = rdt + \sigma dz_t \tag{3}$$

where $dz_t$ is a Wiener process and $\sigma$ is the volatility of the process. The solution to this equation is given by:

$$S_T = S_0 \exp \left( (r - \frac{\sigma^2}{2})T + \sigma z_T \right) \tag{4}$$

Therefore, we can generate a possible trajectory by using:

$$S_{t+\Delta t} = S_t \exp \left( (r - \frac{\sigma^2}{2})\Delta t + \sigma \sqrt{\Delta t} \epsilon \right) \tag{5}$$

where $\epsilon$ is a random variable following a standard normal distribution.

For each given trajectory it is possible to compute the value of the derivative $X(S)$ and therefore we can estimate the fair value of the option and the error associated with this estimate in the standard manner:

$$V_{0,i} = \exp(-rT)\, X(S_i) \tag{6}$$

$$\hat{V_0} = \frac{1}{N} \sum_{i=1}^{N} V_{0,i} \tag{7}$$

and

$$\varepsilon = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} \left(V_{0,i} - \hat{V_0}\right)^2} \tag{8}$$

where $S_i$ is a given trajectory for the underlying asset and $N$ is the number of trajectories generated. In figure 1 we show an example of 50 trajectories generated with $r = 0.1$, $\sigma = 0.2$ and $\Delta t = T/100$, with $T = 1$, all starting with $S_0 = 100$.

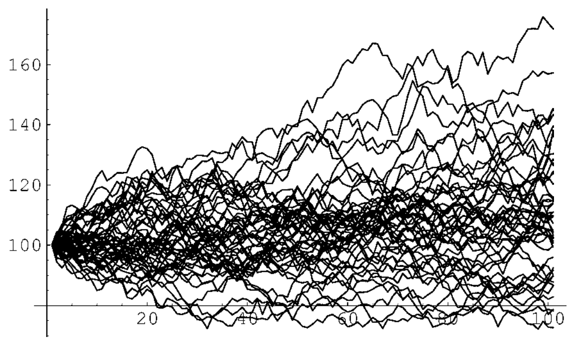


FIG. 1. Fifty paths with 100 steps each for the price of a stock starting with value $S_0 = 100$.

## III. EUROPEAN OPTION

In this section we will study Monte Carlo methods for pricing European options, where $X(S) = \mathrm{Max}(S_T - K, 0)$. In this case we can use the analytical Black-Scholes formula in order to find the exact result and gauge the accuracy of the method when improved with various variance reduction techniques. These techniques can then be used in the precification of more complex option where an analytical pricing formula is not available. We choose the following set of parameters:

$$S = 100, \quad K = 120, \quad \sigma = 0.20, \quad r = 0.12 \;\text{ and }\; T = 1. \tag{9}$$

The exact result is $V_0 = 5.40094$.

### A. Standard Monte Carlo

Using standard Monte Carlo there are two parameters that can be varied: the number of time steps (M) and the number of trajectories (N). In the table below we give the results for the estimate of the option value as well as the error in the estimation for different values of M and N:

| (M,N) | $\hat{V}_0$ | $\varepsilon$ |
|---|---|---|
| (1,1000) | 5.30231 | 0.334931 |
| (1,4000) | 5.48285 | 0.177606 |
| (1,16000) | 5.37964 | 0.0877484 |
| (10,1000) | 5.68429 | 0.347859 |
| (10,4000) | 5.26854 | 0.175044 |
| (10,16000) | 5.35697 | 0.0867721 |

Table I- Estimate of the option value with respective error for different values of numbers of steps (M) and number of trajectories (N).

From this table we can draw two conclusions:

- the number of time steps is not relevant in this case. This is because in this case the option values depends only on the final value of the asset;
- the error of the estimation decreases as $\varepsilon \propto \frac{1}{\sqrt{N}}$, as expected for a statistical error.

### B. Monte Carlo with antithetic variables

The use of two random variables with the same average but opposite correlations, called antithetic variables, is an useful trick to reduce the error (variance) in a Monte Carlo estimation. They can be easily introduced in our calculation by generating two processes for the asset price:

$$S^{(1)}_{t+\Delta t} = S^{(1)}_t \exp\left((r - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}\epsilon\right) \tag{10}$$

and

$$S^{(2)}_{t+\Delta t} = S^{(2)}_t \exp\left((r - \frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t}\epsilon\right) \tag{11}$$

Notice that the only difference is the sign of the volatility term in the exponential. We can generate $N$ trajectories of each type at no extra computer cost and the valuation follows as in equations [7] and [8] but with the difference:

$$V_{0,i} = \exp\left(-rT\right) \frac{X(S^{(1)}_i) + X(S^{(2)}_i)}{2} \tag{12}$$

In the table below we give the results for the estimate of the option value as well as the error in the estimation using antithetic variables for different values of number of time steps (M) and number of trajectories (N):

| (M,N) | $\hat{V}_0$ | $\varepsilon$ |
| --- | --- | --- |
| (1,1000) | 5.04743 | 0.204227 |
| (1,4000) | 5.47227 | 0.110422 |
| (1,16000) | 5.34731 | 0.0537641 |
| (10,1000) | 5.1041 | 0.206186 |
| (10,4000) | 5.16634 | 0.104807 |
| (10,16000) | 5.32275 | 0.0537748 |

Table II- Estimate of the option value with respective error for different values of numbers of steps (M) and number of trajectories (N) using antithetic variables.

The conclusions from the previous subsection are also valid here, but it is important to notice the reduction in the error by a factor of roughly 30% with respect to standard Monte Carlo, which can be relevant in order to reduce computing time to get the same accuracy.

### C. Quasi Monte Carlo

The random numbers generated by the usual algorithms do not distribute themselves in an even fashion. It has been shown that numbers generated by a low discrepancy series, which we call quasi random numbers, are more uniformly distributed. In figure 2 we show the distributions for 400 usual random numbers and for numbers generated by a Holton series.

a) b)

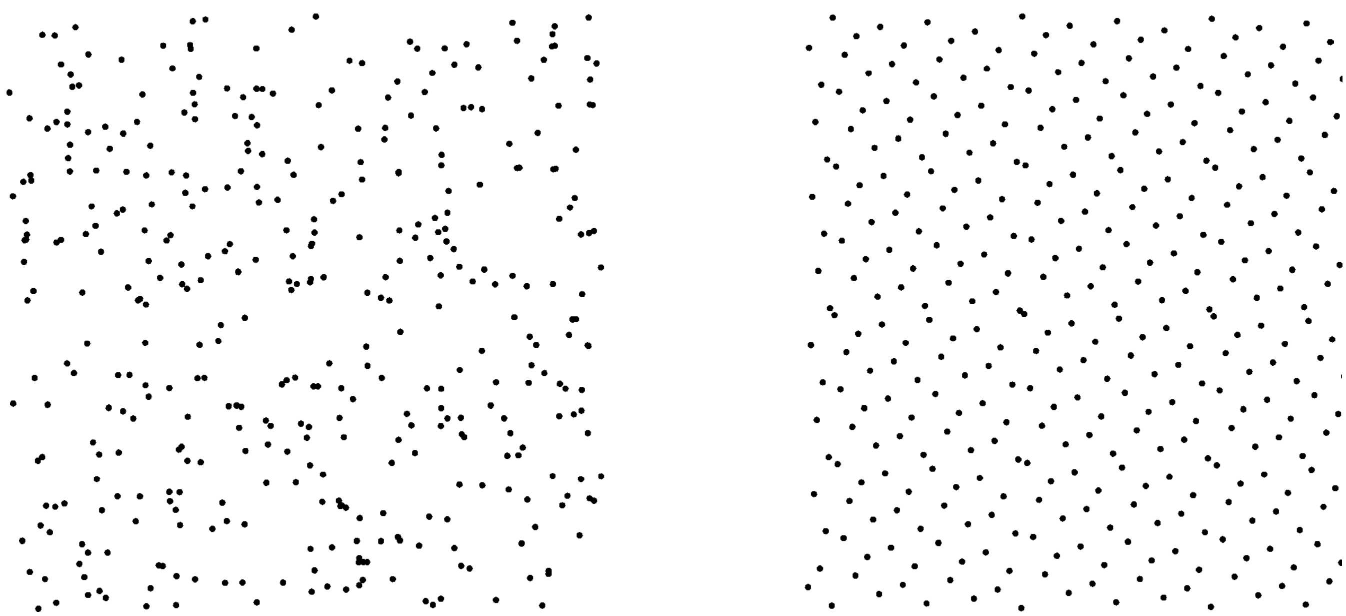

FIG. 2. Distributions of random points in a square for a random number generator (a) and quasi-random number generator (b).

It has been shown that the use of quasi random numbers in Monte Carlo methods, called Quasi Monte Carlo, improves the convergence resulting in a error scaling as $\varepsilon \propto \frac{1}{N}$ as opposed to the usual behaviour $\varepsilon \propto \frac{1}{\sqrt{N}}$ [3].

In the table below we give the results for the estimate of the option value as well as the error in the estimation using quasi random variables for different values of number of time steps (M) and number of trajectories (N):

| (M,N) | $\hat{V}_0$ | $\varepsilon$ |
|---|---|---|
| (1,1000) | 5.32335 | 0.339732 |
| (1,4000) | 5.37793 | 0.173085 |
| (1,16000) | 5.39445 | 0.0871036 |
| (10,1000) | 5.1041 | 0.206186 |
| (10,4000) | 5.16634 | 0.104807 |
| (10,16000) | 5.32275 | 0.0537748 |

Table III- Estimate of the option value with respective error for different values of numbers of steps (M) and number of trajectories (N) using quasi-random variables.

### D. Monte Carlo with Control Variates

Control variates is another method to reduce the variance of a Monte Carlo simulation for option pricing [4]. However, instead of adding an anticorrelated process to the original asset price process, as in the case of antithetic variables, here we will add an anticorrelated process directly to the option price process in order to reduce its variance.

In order to understand the importance of this method, we show in figures 3a and 3b the pay-off and its probability distribution for an european call option. The large width of the probability distribution for the pay-off is responsible for the large error in its evaluation, since the fair price is the discounted value of the its average. The idea of control variates is to reduce this large width, therefore decreasing the error in the option precification.

In this subsection we will study the case of using the delta hedging of the option as a control variate. The Black-Scholes model works by constructing a self-financing replicating portfolio that ideally reproduces the price of the option, whatever happens to the asset process. This portfolio has to be rebalanced (in principle continously) at no cost (neglecting transaction costs) in order for the model to work. The amount of assets in the portfolio in a given instant $t_i$ is given by $\frac{\partial V_{t_i}}{\partial s}$, where $s$ is the asset price at the instant $t_i$. This can be written as:

$$V_T = V_{t_0}\, e^{r(T-t_0)} - \left[\sum_{i=0}^{N}\left(\frac{\partial V_{t_i}}{\partial s} - \frac{\partial V_{t_{i-1}}}{\partial s}\right) S_{t_i} e^{r(T-t_i)}\right] \tag{13}$$

where the expiration date is $T = t_N$ and $\frac{\partial V_{t_N}}{\partial s} = 0$, which means that in the last instant the positions are transfomed in cash without rebalancing. Equation [13] can be re-arranged as:

$$V_T = V_{t_0}\, e^{r(T-t_0)} + \left[\sum_{i=0}^{N-1}\frac{\partial V_{t_i}}{\partial s}\left(S_{t_{i+1}} - S_{t_i} e^{r\Delta t}\right) e^{r(T-t_{i+1})}\right] \tag{14}$$

The quantity in square brackets has zero mean since $E[S_{t_{i+1}}] = S_{t_i} e^{r\Delta t}$ and therefore we can use the second term in equation[14] as an unbiased control variate.

Ideally one would get the hedge variable from the Monte Carlo simulation but here we can use the delta hedge from the analytical Black-Scholes formula.

In the table below we give the results for the estimate of the option value as well as the error in the estimation using the delta hedge control variate for different values of number of time steps (M) and number of trajectories (N):

| (M,N) | $\hat{V_0}$ | $\varepsilon$ |
|---|---|---|
| (1,1000) | 5.29442 | 0.195953 |
| (1,4000) | 5.47534 | 0.0976155 |
| (1,16000) | 5.43094 | 0.0481276 (125.79 Second) |
| (10,1000) | 5.6783 | 0.279954 (115.51 Second) |
| (10,4000) | 5.33241 | 0.134613 (460.1 Second) |

Table IV- Estimate of the option value with respective error for different values of numbers of steps (M) and number of trajectories (N) using delta hedge control variates.

One can notice an almost 50% reduction in the error in comparison with the standard Monte Carlo result, with the same conclusion that adding more intermediate steps does not improve the results.

## IV. BARRIER OPTIONS

### A. Pricing

In this subsection we study the pricing of a down-and-out barrier option with discrete checking points, where no exact analytical solution is available [5]. We use the Monte Carlo method, which can be easily implemented in this case. However, in its simplest form, the method can be very computer time demanding and we will explore some variance reduction techniques developed in the previous section in order to achieve a reasonable accuracy with less computer time. We have checked our codes with the results available in Andersen and Ratcliffe [6] and Gruber [7].

For our studies we concentrate in a down-and-out barrier option with the following parameters:

$$S_0 = 100, \quad K = 90, \quad B = 92, \quad \sigma = 0.2, \quad r = 0.1, \quad \text{and} \quad T = 1 \tag{15}$$

with no dividends, with the barrier checked 50 times.

In the table below we present our results with the corresponding errors and computing time. All the calculations were performed in a Sun Ultra Enterprise 2 workstation using Mathematica 3.0.

| Method | Number of paths | $\hat{V}_0$ | $\varepsilon$ | time (seconds) |
|---|---|---|---|---|
| Standard | 1000 | 15.272 | 0.607621 | 6.64 |
| Standard | 10000 | 15.337 | 0.195324 | 65.6 |
| Antithetic | 1000 | 14.8719 | 0.287694 | 14.9 |
| Antithetic | 10000 | 15.4716 | 0.0896986 | 147 |
| Quasi Monte Carlo | 1000 | 15.1965 | 0.539869 | 92.35 |
| Quasi Monte Carlo | 10000 | 15.6626 | 0.182435 | 950 |
| Control Variate | 1000 | 15.4399 | 0.10492 | 546 |

Table V- Comparison of estimates of the option value with respective error for different values of numbers of steps (M) and number of trajectories (N) using different variance reduction techniques.

For the control variate we use the exact closed-form delta-hedge of a continuously checked barrier option. This is the reason why the computing time for the control variate Monte Carlo is so large, making it impractical for realistic computations. One can also see that due to the high dimensionality of the integration, the Quasi-Monte Carlo method does not provide a good accuracy in comparison with standard Monte Carlo. Therefore, we conclude that the use of antithetic variables provides the best variance reduction technique given the computer time constraint.

### B. Hedging

Besides pricing, it is also very important to develop techniques to compute hedging parameters for barrier options with discrete checking times.

In figures 3–6 we show the exact (continuos checking) results for the pricing (figure 3) and hedging parameters $\Delta$ (figure 4), Vega (figure 5) and $\Gamma$ (figure 6) as a function of the volatility $\sigma$ and of initial stock price $S$ for T = 1 year, interest rates of 10%, no dividends, for four different cases:

**a.** S=100 K=90 B=92 down and out

**b.** S=100 K=110 B=92 down and out

**c.** S=91 K=90 B=92 up and out

**d.** S=80 K=80 B=92 up and out

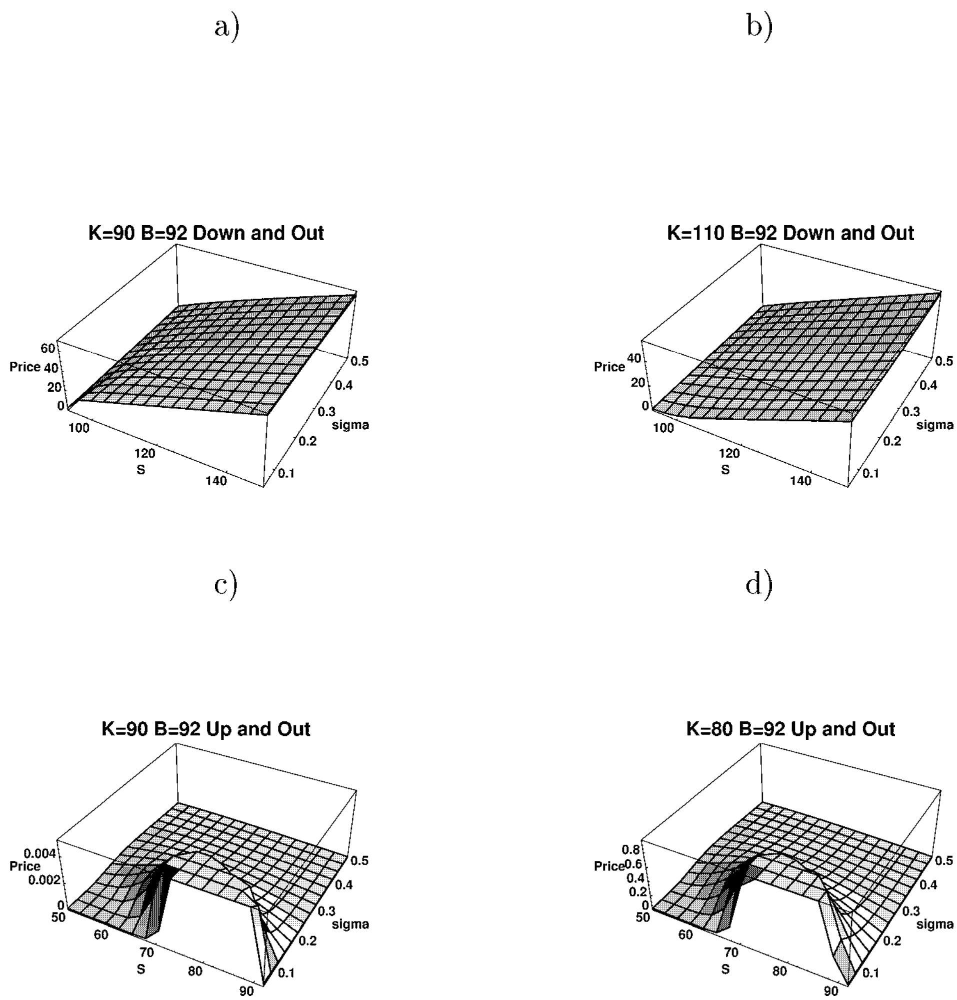


FIG. 3. Price of an european barrier option as a function of initial stock price and volatility.

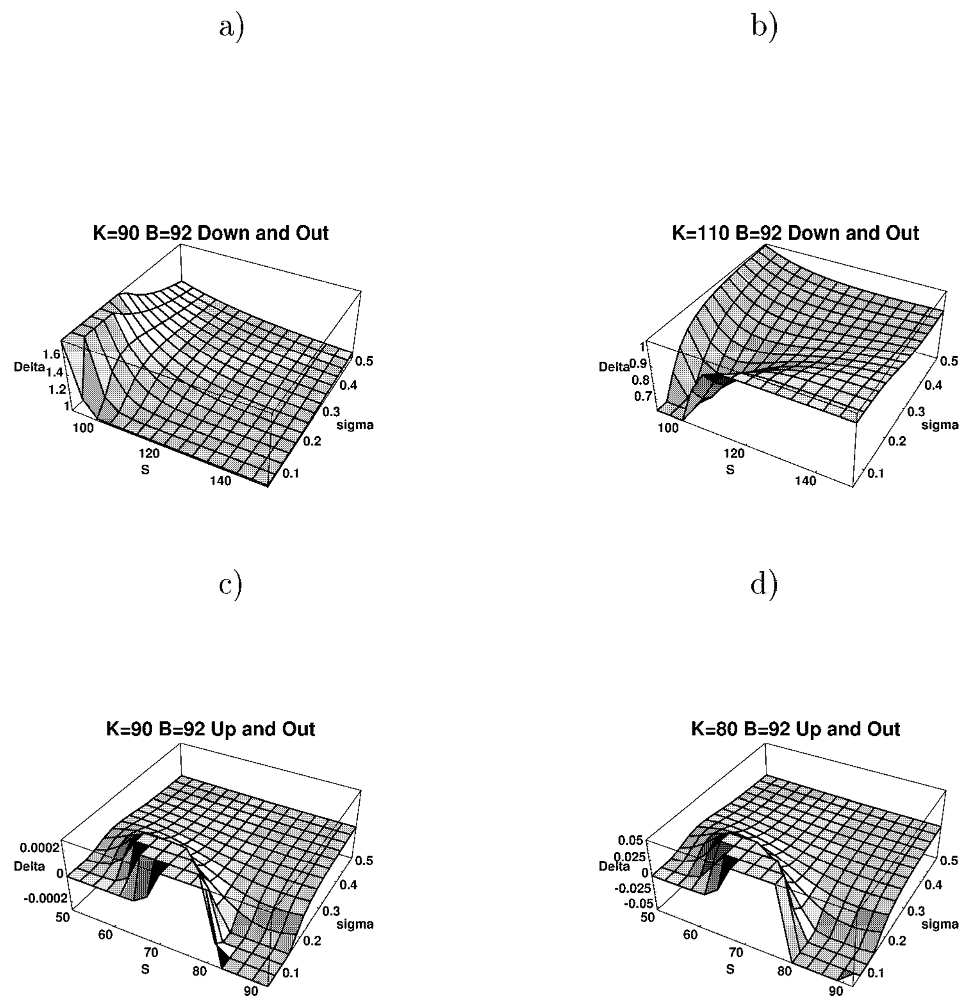


FIG. 4. Delta of an european barrier option as a function of initial stock price and volatility.

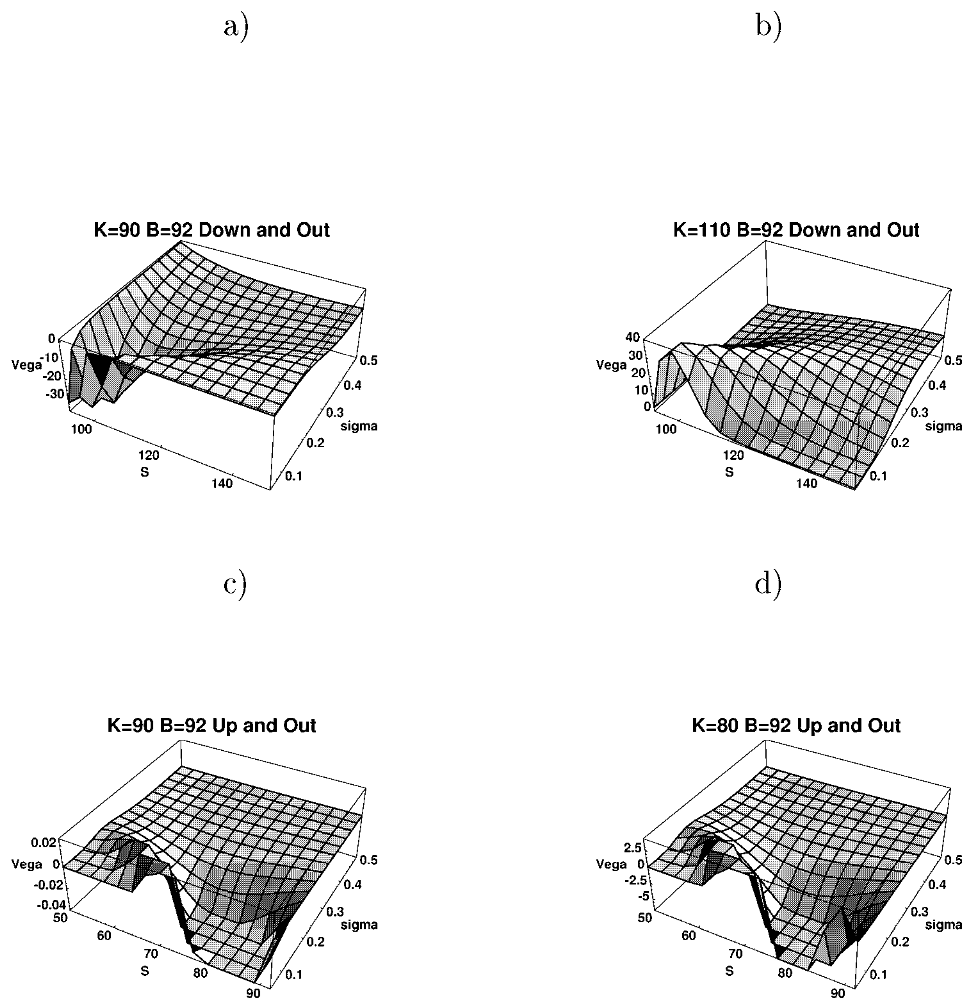


FIG. 5. Vega of an european barrier option as a function of initial stock price and volatility.

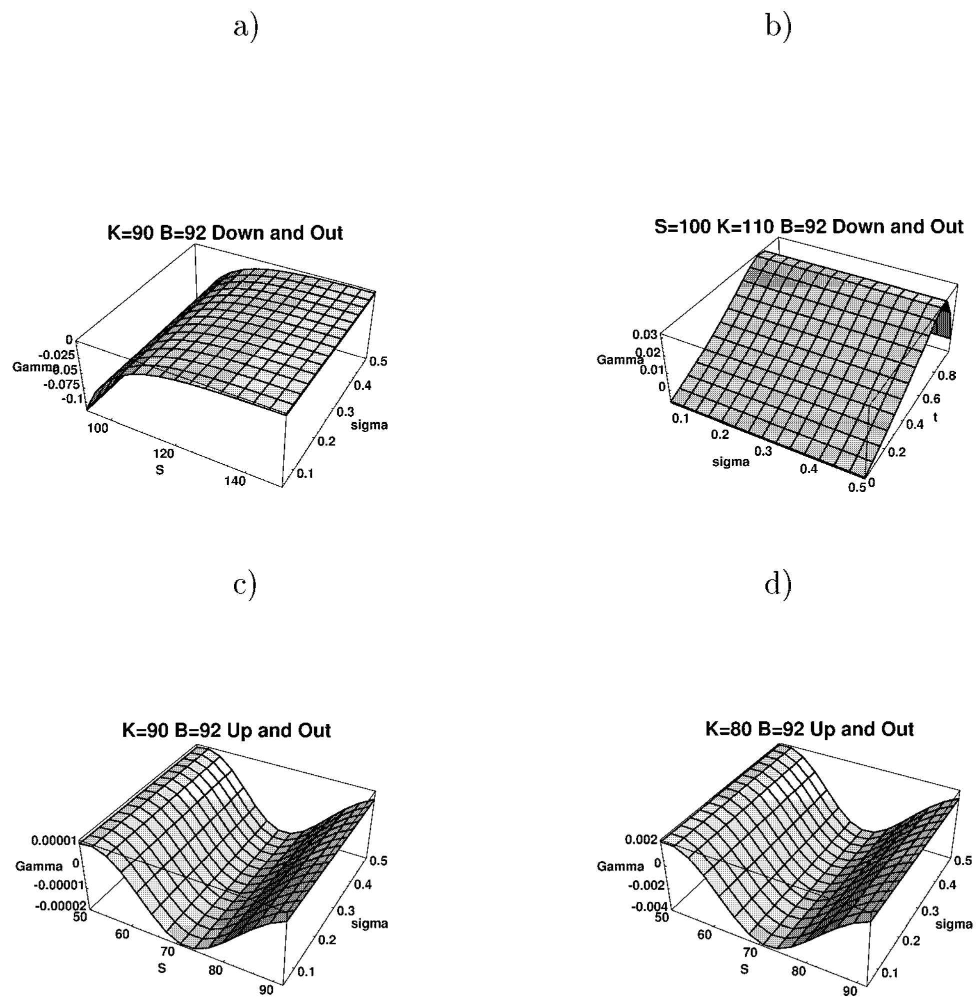


FIG. 6. Gamma of an european barrier option as a function of initial stock price and volatility.

We notice there are large variations in the hedge parameters in the case where the stock price is close to the barrier. The case of an up-and-out option is more striking since there is a competition between the barrier and the strike price since the value of the option increases at the same time that the probability of knocking out also increases. This is a dangerous situation and the headge parameters tend to vary drastically in this situation, as can be seen in the figures above [8].

We are going to use Monte Carlo techniques to calculate hedge parameters for barrier optios with discrete monitoring times, where a closed formula is not available. However, before doing so we test our method with a vanilla call option, where an exact solution is available. We show in the table below a comparison between a standard Monte Carlo calculation with 5.000 paths and the exact results:

| $S_0 = 100,\ K = 120,\ \sigma = 0.2,\ r = 0.12,\ T = 1$ | Exact | Monte Carlo |
|---|---|---|
| Option value | 5.40094 | 5.4 |
| Delta | 0.416207 | 0.42 |
| Gamma | 0.0195055 | 0.021 |
| $S_0 = 120,\ K = 100,\ \sigma = 0.2,\ r = 0.12,\ T = 1$ | Exact | Monte Carlo |
| Option value | 31.8947 | 32 |
| Delta | 0.946476 | 0.95 |
| Gamma | 0.00453635 | -0.000639488 |

Table VI- Estimate of the option value and its delta and gamma hedge parameters for vanilla call options compared with exact results.

We can see that the standard Monte Carlo works well except when gamma is very small, typically less than 0.01, when the results can not be trusted. Fortunately, in this range gamma hedging is not important.

In the tables below we show our results for the hedging variables computed with standard Monte Carlo with 5.000 paths with no dividends, with the barrier checked 10 and 50 times and the exact results from continuous monitoring.

We choose 2 cases:

a) down-and-out barrier option with the following parameters:

$$S_0 = 100, \quad K = 90, \quad B = 92, \quad \sigma = 0.2, \quad r = 0.1, \quad \text{and} \quad T = 1 \tag{16}$$

| $S_0 = 100, \quad K = 90, \quad B = 92$ | Continuous | 50 | 10 |
|---|---|---|---|
| Option value | 14.0153 | 15.6 | 16.9 |
| Delta | 1.48217 | 0.566 | 0.640 |
| Gamma | -0.0502773 | -0.0003 | -0.0013 |

Table VII- Estimate of a down-and-out barrier option value and its delta and gamma hedge parameters for 10 and 50 monitoring times as well as the exact result for continuous monitoring.

b) up-and-out barrier option with the following parameters:

$$S_0 = 80, \quad K = 90, \quad B = 92, \quad \sigma = 0.2, \quad r = 0.1, \quad \text{and} \quad T = 1 \tag{17}$$

| $S_0 = 80, \quad K = 90, \quad B = 92$ | Continuous | 50 | 10 |
|---|---|---|---|
| Option value | 0.00188693 | 0.0056 | 0.014 |
| Delta | -0.000133621 | 0.0072 | 0.017 |
| Gamma | -0.0000113231 | – | – |

Table VIII- Estimate of an up-and-out barrier option value and its delta and gamma hedge parameters for 10 and 50 monitoring times as well as the exact result for continuous monitoring.

A statistical error for any quantity $V$ estimated via Monte Carlo of $V/\sqrt{5000}$ has to be considered. As expected, the price of the barrier option decreases with the increase of the frequency of observation, being minimum in the continuum case. Also notice the difference in the hedge parameters from the continuous case in the more realistic case of discrete observation times. This difference is even more striking in the second case, when the stock price is very close to the barrier and hence the option price is small.

## V. PRICING AND HEDGING BARRIER OPTIONS WITH A VOLATILITY TERM STRUCTURE

The values of the hedging parameters depend on the volatility of the stock, which was assumed to be constant so far. However, it is known that in realistic markets the volatilities are not constant. We have also seen in the figures of the last section how large is the sensitivity to the volatility for certain hedge parameters. In this section we will study the impact of different volatility term structures on the pricing and hedging of barrier options. We will divide the time to expiration in three and study three different scenarios for the volatility in these intervals:

a) $\sigma_1 = 0.2$, $\sigma_2 = 0.2$ and $\sigma_3 = 0.2$

b) $\sigma_1 = 0.05$, $\sigma_2 = 0.2$ and $\sigma_3 = 0.7$

c) $\sigma_1 = 0.7$, $\sigma_2 = 0.2$ and $\sigma_3 = 0.05$

Our results, again for standard Monte Carlo with 5.000 paths with no dividends, with the barrier checked 10 times are:

| Down-and-Out($S_0 = 100,\ \ K = 90,\ \ B = 92$) | (a) | (b) | (c) |
|---|---|---|---|
| Option value | 16.9 | 21.6 | 20.4 |
| Delta | 0.640 | 0.517 | 0.502 |

Table IX- Estimate of a down-and-out barrier option value and its delta hedge parameter for different volatility scenarios for 10 checking times.

We notice an increase in the price for the non-constant volatilities. The price depends on the volatility structure, being lower for an initial high volatility period, as expected. The deltas also changed with respect to the constant volatility case but no difference is seen between the different volatility structures.

| Up-and-Out($S_0 = 80,\ \ K = 90,\ \ B = 92$) | (a) | (b) | (c) |
|---|---|---|---|
| Option value | 0.014 | 0.0058 | 0.0057 |
| Delta | 0.017 | 0.0066 | 0.0072 |

Table X- Estimate of an up-and-out barrier option value and its delta hedge parameter for different volatility scenarios for 10 checking times.

We notice a decrease in the price for the non-constant volatilities and in this case the prices don't depend on the volatility structure. The deltas also changed with respect to the constant volatility case but no statistically significant difference is seen between the different volatility structures.

## VI. CONCLUSIONS

In the first part of the paper we studied several variance reduction techniques for Monte Carlo estimation of prices for European barrier options with discrete time monitoring. We concluded that the best compromise between error reduction and computing time is achieved by the use of atithetic variables.

In the second part of the paper we computed in a couple of examples hedge parameters and studied the impact of a volatility term structure on the pricing and hedging of barrier options. In this case, there are no approximate formulas in the literature and therefore Monte Carlo simulation becomes a very important tool.

The codes developed in this work are available upon request from `rosenfel@ift.unesp.br`.

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